5. (15 Points) Let $f \in Isom(\mathbb{R}^2)$ be a non-trivial rotation about a point $\vec{p}$ and $g \in Isom(\mathbb{R}^2)$ be a non-trivial rotation about a point $\vec{q}$. Show that $fg = gf$ if and only if $\vec{p} = \vec{q}$.

Clearly, if $\vec{p} = \vec{q}$ then $fg = gf$.

Suppose $fg = gf$.

Then $(fg)(\vec{q}) = f(g(\vec{q})) = f(\vec{q})$

$= (gf)(\vec{q}) = g(f(\vec{q}))$

$\to f(\vec{q})$ is fixed by g

$\Rightarrow f(\vec{q}) = \vec{q}$

$\Rightarrow \vec{q} = \vec{p}$.

6. (15 Points) Describe finite subgroups of $Isom(\mathbb{R})$.

Recall $Isom(\mathbb{R}) = T \rtimes \langle s \rangle$

where $T = \{T_x \mid x \in \mathbb{R}\}$ $\quad s(x) = -x \ \forall x$.

If $G \leq Isom(\mathbb{R})$ is finite

then G has no non-trivial translations.

if $T_x \cdot s, T_y \cdot s \in G$ then

$T_x \cdot s T_y \cdot s = T_{x-y} \in G \Rightarrow x = y$.

So, $G = \{e\}$ or $G = \{e, T_x \cdot s\} \cong \mathbb{Z}_2$

for some $x \in \mathbb{R}$.

M E T U Department of Mathematics

Math 466 - Spring 2025 - Midterm 1 - April 14 - 14:40 - Instructor: Gökhan Benli		
FULL NAME (write in CAPITAL letters) EMMY NOETHER	STUDENT ID	SIGNATURE
6 QUESTIONS ON 4 PAGES	110 MINUTES	TOTAL 100 POINTS

1. (20 Points) let D_n $n \geq 3$ be the dihedral group of order $2n$. Show that the center of D_n is trivial if n is odd and contains two elements if n is even. (If G is a group, its *center* is $Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}$)

Recall $D_n = \{ r^i s^j \mid i \in \{0, .., n-1\}, j \in \{0, 1\} \}$

where $r(x) = x+1 \ \forall x \in \mathbb{R}$.
$s(x) = x \ \forall x \in \mathbb{R}$.

Note that $sr = r^{-1}s \neq rs$ since $r \neq r^{-1}$ ($n \geq 3$!)

So r and s are not in $Z(D_n)$ if $n \geq 3$.

Also if $r^i s \in D_n$ with $i \in \{1, .., n-1\}$

$(r^i s) \cdot r = r(r^i s) \iff r^{i-1} = r^{i+1} \iff r \neq r^{-1}$

So, $r^i s \notin Z(D_n)$ with $i \in \{1, .., n-1\}$ for $n \geq 3$.

For $i \in \{1, .., n-1\}$, $r^i s = s r^i \iff r^{2i} = e$
$\iff n \mid 2i, \ 2i \in \{2, .., 2n-2\}$
$\iff n = 2i$

So, if n is odd $r^i \notin Z(D_n) \ \forall i = 1, .., n-1$

$\Rightarrow Z(D_n) = \{e\}$

if n is even, $(r^i s) \cdot r^{n/2} = r^i r^{n/2} s = r^{n/2}(r^i s)$

so $r^{n/2} \in Z(D_n)$. Thus $Z(D_n) = \{e, r^{n/2}\}$

2. (30 pts) In each of the following cases, describe the given element of $Isom(\mathbb{R}^2)$. (i.e., state whether it is a translation, rotation, reflection or glide reflection and give necessary information such as rotation center, reflection line etc.)

(a) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \frac{\pi}{2} \end{bmatrix} \Rightarrow$ rotation about $(0,0)$ by $\pi/2$.

(b) $f = T_{\vec{u}} \cdot A$ where $\vec{u} = (1, 2)$.

$(I - A)\vec{p} = \vec{u} \iff \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \iff \begin{matrix} p_1 + p_2 = 1 \\ -p_1 + p_2 = 2 \end{matrix}$

$\iff \vec{p} = \begin{bmatrix} -1/2 \\ 3/2 \end{bmatrix}$

f is rot. about $\vec{p}$ by $\frac{\pi}{2}$.

(c) $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \cos -\frac{\pi}{2} & \sin(-\frac{\pi}{2}) \\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{bmatrix} \Rightarrow$ reflection wrt $y = \tan(-\frac{\pi}{4})x$

ie $y = -x$

(d) $g = T_{\vec{v}} \cdot B$ where $\vec{v} = (1, 1)$

B is reflection wrt $l: y = -x$ and $\vec{u} \perp l$.

So g is a reflection.

need $\vec{p}$ s.t $\vec{p} \in l^{\perp}$ and $(I - B)\vec{p} = \vec{v}$

$\iff \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $p_1 = p_2 \iff \vec{p} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

So, g is reflection wrt $T_{\vec{p}}(l) : y = -x + 1$

3. (10 Points) Let $T_{\vec{v}}$ and $T_{\vec{w}}$ be two translations in $Isom(\mathbb{R}^2)$. Show that they are conjugate if and only if $||\vec{v}|| = ||\vec{w}||$.

let $g = T_{\vec{u}} \cdot A$

$$g\, T_{\vec{v}} \cdot g^{-1} = T_{\vec{w}}$$

$$\Leftrightarrow \quad T_{\vec{u}} \cdot A \cdot T_{\vec{v}} \cdot A^{-1} \cdot T_{-\vec{u}} = T_{\vec{w}}$$

$$\Leftrightarrow \quad T_{\vec{u} + A\vec{v}} = T_{\vec{w} + \vec{u}}$$

$$\Leftrightarrow \quad \vec{u} + A\vec{v} = \vec{w} + \vec{u}$$

$$\Leftrightarrow \quad A\vec{v} = \vec{w}$$

So, if $T_{\vec{v}}$ and $T_{\vec{w}}$ are conjugate then $A\vec{v} = \vec{w} \Rightarrow ||\vec{v}|| = ||\vec{w}||$.

Conversely, if $||\vec{u}|| = ||\vec{v}||$ let $A \in O(2)$ be the rotation s.t $A\vec{u} = \vec{v}$. hence $A \cdot T_{\vec{v}} \cdot A^{-1} = T_{A\vec{u}} = T_{\vec{w}}$.

4. (10 Points) Let K and H be groups. Show that if $\varphi : K \to Aut(H)$ is a non-trivial group homomorphism, then $H \rtimes_\varphi K$ is a non-abelian group.

let $k \in K$ st $\varphi(k) \neq id_H$. So, $\exists h \in H$ with $\varphi(k)(h) \neq h$.

then

$$(1, k)(h, 1) = (\varphi(k)(h), k) \neq (h, k) = (h, 1)(k, 1).$$

So, $H \rtimes_\varphi K$ is non-abelian.