Math 466 Exercises for Week 4

March 13, 2025

1. Following the steps below prove the following:

   Let $U \subseteq \mathbb{R}^n$ and suppose $f : U \to \mathbb{R}^n$ is distance preserving. Then there exists an isometry $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\hat{f}\mid_U = f$.

   (a) If $u \in U$, then $g = T_{-f(u)} \cdot f \cdot T_u : T_{-u}(U) \to \mathbb{R}^n$ is distance preserving and fixes the origin. Thus WLOG we may assume U contains the origin and f fixes the origin.

   (b) Show that $f(x) \cdot f(y) = x \cdot y$ for all $x, y \in U$.

   (c) Let $V = \langle U \rangle$ be the subspace spanned by U. Given $v \in V$, write $v = \sum_{i=1}^{n} c_i u_i$ for some $u_i \in U$ and define $\hat{f} : V \to \mathbb{R}^n$ by $\hat{f}(v) = \sum_{i=1}^{n} c_i f(u_i)$. Show that $\hat{f}$ is well defined, distance preserving linear transformation.

   (d) Let $W = \langle f(U) \rangle$ and note that V and W have the same dimension. Find a linear isometry $\bar{f} : \mathbb{R}^n \to \mathbb{R}^n$ which extends $\hat{f}$. (Hint: Use orthogonal complements.)

2. Recall that, given an isometry $f \in Isom(\mathbb{R}^2)$, we write $f = T_{\vec{v}} \cdot A$ where $T_{\vec{v}}$ is a translation and $A \in O(2)$ is an orthogonal matrix. Then, if is O.P. if $det(A) = 1$ an O.R. otherwise. $Isom^+(\mathbb{R}^2)$ is the subgroup of $Isom(\mathbb{R}^2)$ consisting of orientation preserving isometries. L

   Show that $Isom^+(\mathbb{R}^2) \cong \mathbb{R}^2 \rtimes SO(2)$.

3. Show that the square of an O.R. isometry of $\mathbb{R}^n$ is a translation.

For the remaining exercises, all elements belong to $Isom(\mathbb{R}^2)$.

4. Show that any glide reflecton can be written as the product of a reflection and a rotation. Is this decomposition unique?

5. Let f and g be glide reflections about the lines ℓ_1 and ℓ_2 respectively. Show that $f \cdot g$ is a translation if and only if $\ell_1 \parallel \ell_2$.

6. Prove that any two reflections are conjugate.

7. Let f, g be rotations by θ_1 and θ_2 where $\theta_i \in (0, 2\pi)$. Show that f and g are conjugate if and only if $\theta_1 = \theta_2$.

8. Let $T_{\vec{v}}$ and $T_{\vec{w}}$ be two translations. Show that they are conjugate if and only if $||\vec{v}|| = ||\vec{w}||$.