Math 466 Exercises for Week 3

March 9, 2025

1. Let $\vec{u} \in \mathbb{R}^n$ be a unit vector and $\lambda \in \mathbb{R}$.

   (a) Let $P = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{u} = \lambda\}$ and $W = \langle \vec{u} \rangle^{\perp}$. Show that $P = T_{\lambda \vec{u}}(W)$.

   (b) Let $R : \mathbb{R}^n \to \mathbb{R}^n$ be the reflection w.r.t P given by $R(\vec{x}) = \vec{x} + 2(\lambda - \vec{x} \cdot \vec{u})\vec{u}$. Show that $R \in Isom(\mathbb{R}^n)$.

2. Show that if $A \in O(n)$ then $det(A) = \pm 1$. Show that the converse is not true.

3. (a) Let $SO(n) = \{A \in O(n) \mid det(A) = 1\}$ is a normal subgroup of $O(n)$. (This is the "special orthogonal group".) What is the quotient group $O(n)/SO(n)$ isomorphic to?

   (b) Find a matrix $A \in O(n) - SO(n)$ such that $K = \{I, A\}$ is a subgroup of $O(n)$ and $O(n) \cong SO(n) \rtimes K$

   (c) If n is odd, find a matrix $A \in O(n) - SO(n)$ such that $K = \{I, A\}$ is a subgroup of $O(n)$ and $O(n) \cong SO(n) \times K$

4. (Some elementary Euclidean geometry) Recall that two triangles are called *congruent* if their vertices can be labelled as A, B, C and A', B', C' such that the corresponding edges and angles are equal.

   Also recall a result from elementary geometry that if two triangles have equal side lengths, then they are congruent.

   Using these only, show that if $f \in Isom(\mathbb{R}^2)$ then

   (a) f maps congruent triangles to congruent triangles

   (b) f preserves angles

   (c) f maps lines to lines

   (d) f is uniquely determined by its values on three non-collinear (meaning not on a line) points.