Math 466 Exercises for Week 2

February 27, 2025

1. Let $x, y \in \mathbb{R}$. Show that there are exactly two isometries of $\mathbb{R}$ which send x to y.

   Recall that two elements a, b in a group G are conjugate if there is $g \in G$ such that $a = gbg^{-1}$. (This is an equivalence relation on G whose equivalence classes are called *conjugacy classes*).

2. Describe the conjugacy classes of $Isom(\mathbb{R})$.

3. Let G be a group. Show that the function $f(x) = x^{-1}$ is an automorphism of G if and only if G is abelian.

4. Let H, K be groups and $\varphi : K \to Aut(H)$ be a homomorphism. Define a binary operation on $H \times K$ by

   $$(h_1, k_1)(h_2, k_2) = (h_1\varphi(k_1)(h_2), k_1k_2)$$

   Show that this operation is associative.

5. Show that if $\varphi : K \to Aut(H)$ is non-trivial, then $H \rtimes_\varphi K$ is non-abelian.

6. Let G be a group and H, K subgroups of G with $H \lhd G$, $H \cap K = \{e\}$ and $G = HK$. Show that $G \cong H \rtimes_\varphi K$ for some homomorphism $\varphi : K \to Aut(H)$.