Math 371 Recitation 3

1.) In which of the following cases is the mapping $x: \mathbb{R}^2 \to \mathbb{R}^3$ a patch

(a) $x(u,v) = (u, u^2, v+v^3)$

Recall: $x: D \to \mathbb{R}^3$ is a patch if $D \subset \mathbb{R}^2$ is open, x is 1-1 and regular.

Say $x(u_1, v_1) = x(u_2, v_2)$. Then $(u_1, u_1^2, v_1+v_1^3) = (u_2, u_2^2, v_2+v_2^3)$ implies $u_1 = u_2$.

Note that $g(v) = v+v^3$ is increasing on $\mathbb{R}$ as $g'(v) = 1+3v^2 > 0$ $\forall v \in \mathbb{R}$

$\Rightarrow g$ is 1-1 on $\mathbb{R}$. Hence,

$$v_1 + v_1^3 = v_2 + v_2^3 \Rightarrow v_1 = v_2$$

and this shows that x is 1-1.

The Jacobian matrix of x is

$$\begin{pmatrix} 1 & 2u & 0 \\ 0 & 0 & 1+3v^2 \end{pmatrix}$$

Since $1+3v^2 \neq 0$, the rows of this matrix are linearly independent.

So, the rank is 2 for each $(u,v) \in \mathbb{R}^2$. Hence $T_{x(u,v)}$ is 1-1 and x is a patch.

(b) $x(u,v) = (u^2, u^3, v)$

Note that the rows of the Jacobian $\begin{pmatrix} 2u, & 3u^2, & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are not linearly independent at $(0,0)$.

Therefore, the mapping is not a patch.

(c) $x(u,v) = (\cos 2\pi u, \sin 2\pi u, v)$

$x(0,0) = (\cos 0, \sin 0, 0) = (1,0,0)$, $x(1,0) = (\cos 2\pi, \sin 2\pi, 0)$
$= (1,0,0)$

So, x is not 1-1 and it can not be a patch

2-) (a) Prove that $M: (x^2+y^2)^2 + 3z^2 = 1$ is a surface.

Recall: If g is any differentiable function on $\mathbb{R}^3$ and $c \in \mathbb{R}$ such that $dg \neq 0$ at any point of $M: g(x,y,z) = c$, then M is a surface.

Define $g(x,y,z) = (x^2+y^2)^2 + 3z^2$ on $\mathbb{R}^3$. Clearly g is differentiable.

$$dg = (2(x^2+y^2)2x)\,dx + (2(x^2+y^2)2y)\,dy + 6z\,dz$$

$$dg = 0 \iff x(x^2+y^2) = y(x^2+y^2) = z = 0$$

$$\iff x=0,\ y=0,\ z=0$$

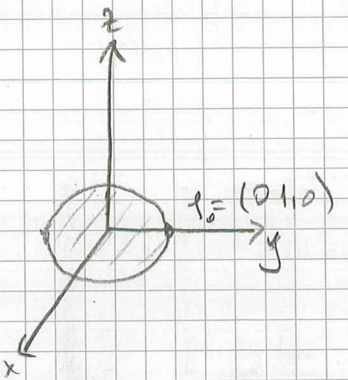
Note that $(0,0,0) \notin M$ as it does not satisfy $(x^2+y^2)^2 + 3z^2 = 1$.

Hence, M is a surface.

(b) For which values of c, is $M: z(z-2) + xy = c$ a surface?

(exercise)

3.) Is the set $M = \{(x,y,z) \in \mathbb{R}^3 \mid z=0 \text{ and } x^2+y^2 \leq 1\}$ a surface?



Say, it is a surface. Then there exists an open $U \subset \mathbb{R}^2$, a neighborhood $V = \{(x,y,z) \in M \mid d((x,y,z), p_0) < \varepsilon\}$ a $\underbrace{\text{proper}}_{\varphi^{-1} \text{ is continuous}}$ patch $\varphi: U \longrightarrow V$.

Note that V can be identified with the set $\{(x,y) \mid x^2+y^2 \leq 1, d((x,y),(0,1)) < \varepsilon\}$ in $\mathbb{R}^2$ which is clearly not an open set.

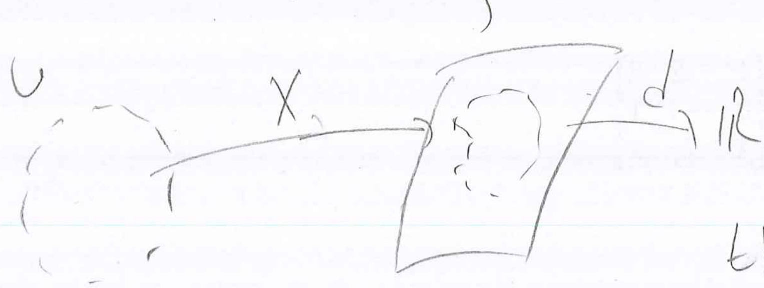
φ induces a homeomorphism between two subsets of $\mathbb{R}^2$ one of them is open and the other is not. This causes a contradiction. Hence, M is not a surface.

What about $M' = \{(x,y,z) \mid z=0, x^2+y^2<1\}$?

Consider $U = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$. U is open in $\mathbb{R}^2$ and $\varphi: U \longrightarrow M'$ defined by $\varphi(x,y) = (x,y,0)$ is clearly a homeomorphism whose Jacobian matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has rank 2.

Therefore, M' is a simple surface as $\varphi(U) = M'$.

4-) Let $S \subset \mathbb{R}^3$ be a surface and $d: S \to \mathbb{R}$ be defined by $d(p) = |p - p_0|$ for $p_0 \in \mathbb{R}^3 \setminus S$. Prove that d is differentiable.



Consider any patch $X: U \subseteq \mathbb{R}^2 \to S \subset \mathbb{R}^3$ such that $X(u,v) = (x(u,v), y(u,v), z(u,v))$.

If $p_0 = (a,b,c)$, then $d \circ X: U \to \mathbb{R}$

$$(u,v) \longmapsto \sqrt{(x(u,v)-a)^2 + (y(u,v)-b)^2 + (z(u,v)-c)^2}$$

Since $p_0 \notin S$ and x, y, z are differentiable functions, $d \circ X$ is differentiable. As the patch X is arbitrary, this make d a differentiable function.

5-) Prove that $v = (v_1, v_2, v_3)$ is tangent to $M: z = f(x,y)$ at $p = (p_1, p_2, f(p_1, p_2))$ iff

$$v_3 = \frac{\partial f}{\partial x}(p_1, p_2) + \frac{\partial f}{\partial y}(p_1, p_2) v_2.$$

Recall: If X is a patch such that $X(u_0, v_0) = p$, then v is tangent to M at p iff v can be written as a linear combination of $X_u(u_0, v_0)$ and $X_v(u_0, v_0)$.

Consider the patch $X(u,v) = (u, v, f(u,v))$ from any ope set in $D(f)$ containing (p_1, p_2).

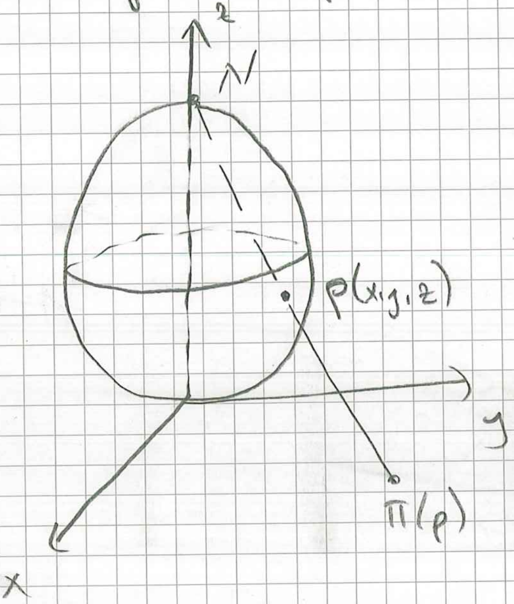
By the recall, $v = (v_1, v_2, v_3)$ is tangent to M at e iff

$$(v_1, v_2, v_3) = c_1(1, 0, f_u(e_1, e_2)) + c_2(0, 1, f_v(e_1, e_2))$$

for some $c_1, c_2 \in \mathbb{R}^*$.

This happens if and only if $v_3 = v_1 f_u(e_1, e_2) + v_2 f_v(e_1, e_2)$.

5-) $S^2: x^2+y^2+(z-1)^2=1$, $N=(0,0,2)$, $\pi: S^2-\{N\}\to\mathbb{R}^2$ which carries $p=(x,y,z)$ onto the intersection of the xy-plane with straight line connecting N to p. Find $\pi^{-1}:\mathbb{R}^2\longrightarrow S^2-\{N\}$



Let $(u,v)\in\mathbb{R}^2$ be any point. Then $\pi^{-1}(u,v)$ is the unique point of intersection of $S^2\setminus\{N\}$ and the line l_{uv} joining N to (u,v). It is parametrized by

$$(x(t),y(t),z(t))=t(u,v,0)+(1-t)(0,0,2)$$
$$t\in[0,1]$$

We must have $(x(t),y(t),z(t))\in S^2$ for some t.

$$t^2u^2+t^2v^2+(1-2t)^2\,4=1 \Rightarrow t^2(u^2+v^2+4)=4t$$

$$(x(t),y(t),z(t))\neq N \Rightarrow t\neq 0 \quad\Rightarrow\quad t=\frac{4}{u^2+v^2+4}$$

Hence, $$\pi^{-1}(u,v)=\left(\frac{4u}{u^2+v^2+4},\frac{4v}{u^2+v^2+4},\frac{2(u^2+v^2)}{u^2+v^2+4}\right),\ (u,v)\in\mathbb{R}^2$$

$t(x,y,z)+(1-t)(0,0,2)$ is the line joining (x,y,z) to N. It intersects with the xy-plane if $tz+(1-t)2=0$, i.e, $t=\dfrac{2}{2-z}$.

Then, $\pi(x,y,z)=\left(\dfrac{2x}{2-z},\dfrac{2y}{2-z},0\right)$. To show: π and π^{-1} are inverses of each other (exercise)

Clearly both π and π^{-1} are differentiable. One can check $\pi^{-1}_{*(u,v)}$ is 1-1 for any $(u,v)\in\mathbb{R}^2$. Hence, π^{-1} is a proper patch for S^2. A similar patch $\varphi:\mathbb{R}^2\to S^2-\{(0,0,0)\}$ can be defined to cover S^2 with only 2 patches.