Math 371 Recitation 2

1.) Let $\alpha: I \to \mathbb{R}^3$ be a parametrized curve. Let $[a,b] \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

(a) Show that, for any constant vector v, $|v| = 1$,

$$(q-p)\cdot v = \int_a^b \alpha'(t)\cdot v\,dt \leq \int_a^b |\alpha'(t)|\,dt$$

Observe that $\alpha(t)\cdot v$ is an anti-derivative for $\alpha'(t)\cdot v$:

$$\frac{d}{dt}(\alpha(t)\cdot v) = \alpha'(t)\cdot v + \alpha(t)\cdot \underbrace{v'}_{=0} = \alpha'(t)\cdot v$$

Therefore,
$$\int_a^b \alpha'(t)\cdot v\,dt = (\alpha(t)\cdot v)\Big|_{t=b} - (\alpha(t)\cdot v)\Big|_{t=a} = (q-p)\cdot v \text{ by FTC}$$

Also $\alpha'(t)\cdot v = \|\alpha'(t)\|\,\underbrace{\|v\|}_{=1}\cos\theta \leq \|\alpha'(t)\|$, where θ is the angle between v and $\alpha'(t)$

$$\Rightarrow (q-p)\cdot v = \int_a^b (\alpha'(t)\cdot v)\,dt \leq \int_a^b \|\alpha'(t)\|\,dt$$

(b) Set $v = \dfrac{q-p}{\|q-p\|}$ and show that $|\alpha(b) - \alpha(a)| \leq \int_a^b \|\alpha'(t)\|\,dt$

By the first part, we obtain

$$(q-p)\cdot v = (q-p)\cdot\frac{(q-p)}{\|q-p\|} \leq \int_a^b \|\alpha'(t)\|\,dt$$

$$\Rightarrow \|\alpha(b) - \alpha(a)\| \leq \int_a^b \|\alpha'(t)\|\,dt;$$ that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these points.

2.) Compute Frenet apparatus κ, τ, T, N, B of the unit speed curve $\beta(s) = (4/5 \cos s, 1-\sin s, -3/5 \cos s)$. Show that it is a circle, find its center and radius.

$$T^{(s)} = \beta' = \left(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\right), \quad T' = \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right)$$

$$\kappa^{(s)} = \|T'\| = \sqrt{\frac{16}{25}\cos^2 s + \sin^2 s + \frac{9}{25}\cos^2 s} = 1$$

$$\Rightarrow N^{(s)} = T'/\kappa = T'$$

$$B^{(s)} = T \wedge N = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\frac{4}{5}\sin s & -\cos s & \frac{3}{5}\sin s \\ -\frac{4}{5}\cos s & \sin s & \frac{3}{5}\cos s \end{vmatrix} = \left(-\frac{3}{5}\cos^2 s - \frac{3}{5}\sin^2 s, 0, -\frac{4}{5}\sin^2 s - \frac{4}{5}\cos^2 s\right)$$

$$= (-3/5, 0, -4/5)$$

$$B' = 0, \quad B' = -\tau N \Rightarrow \tau = 0$$

(since it is closed, it must be the whole circle)

κ is constant and $\tau = 0 \Rightarrow \beta$ is a ~~part of~~ circle of radius $1/\kappa = 1$

$\beta(0) = (-3/5, 0, +4/5)$ lying in $z = -\frac{3}{4}x$

$$\beta + \frac{1}{\kappa}N = \left(\frac{4}{5}\cos s, 1-\sin s, -\frac{3}{5}\cos s\right) + \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right) = (0, 1, 0)$$

is the center

3.) Let $\beta, \bar{\beta} : I \to \mathbb{R}^3$ be unit-speed curves with nonvanishing curvature and torsion.

a) $T = \bar{T} \Rightarrow \beta$ and $\bar{\beta}$ are parallel

$$T = \bar{T} \Rightarrow \beta'(s) - \bar{\beta}'(s) = (\beta(s) - \bar{\beta}(s))' = 0$$

$$\Rightarrow \beta(s) = \bar{\beta}(s) + p \text{ for some fixed } p \in \mathbb{R}^3$$

$$\Rightarrow \beta \text{ and } \bar{\beta} \text{ are parallel}$$

(b) If $B = \bar{B}$, prove that $\bar{\beta}$ is parallel to either β or to the curve $s \longrightarrow -\beta(s)$

$$B = \bar{B} \Rightarrow B' = \bar{B}' \Rightarrow \tau(s)N(s) = \bar{\tau}(s)\bar{N}(s)$$

$$\Rightarrow \quad |\tau| = |\tau|\,|N| = |\bar{\tau}|\,|\bar{N}| = |\bar{\tau}|$$

$$\Rightarrow N = \bar{N},\ \tau = \bar{\tau} \quad \text{or} \quad N = -\bar{N},\ \tau = -\bar{\tau}$$

$N' = -\kappa T + \tau B \quad \Rightarrow \kappa T = \bar{\kappa}\bar{T} \quad$ or $\quad \kappa T = -\bar{\kappa}\bar{T}$

(Taking the norm) $\Rightarrow \kappa = \bar{\kappa}$ as $\kappa > 0$

$$\Rightarrow T = \bar{T} \quad \text{or} \quad T = -\bar{T}$$

By part (a), the proof is over

Let α be a unit-speed curve with $\kappa' > 0$, $\tau \neq 0$.

(a) If α lies on a sphere of center c and radius r, show that

$$\alpha - c = -\rho N - \rho' \sigma B,$$

where $\rho = 1/\kappa$, $\sigma = 1/\tau$. Thus, $r^2 = \rho^2 + (\rho'\sigma)^2$. (exercise)

(b) If $\rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$, show that α lies on a sphere of radius r.

Aim is to prove $\alpha + \rho N + \rho' \sigma B$ is constant.

$$(\alpha + \rho N + \rho' \sigma B)' = T + \rho' N + \rho \underbrace{N'}_{-\kappa T + \tau B} + \rho'' \sigma B + \underbrace{\rho' \sigma' B}_{0} + \rho' \sigma \underbrace{B'}_{-\tau N}$$

$$= T(\underbrace{1 - \rho \kappa}_{0}) + N(\cancel{\rho' - \tau \rho' \sigma}) + B(\rho \tau + \rho'' \sigma + \rho' \sigma') \quad (1)$$

$$\rho^2 + (\rho'\sigma)^2 = r^2 \Rightarrow \qquad 0 = (\rho^2 + (\rho'\sigma)^2)' = 2\rho\rho' + 2(\rho'\sigma).(\rho'\sigma)'$$

$$\Rightarrow \rho'(\rho + \sigma(\rho''\sigma + \rho'\sigma')) = 0 \qquad \text{as } \rho' \neq 0$$

$$\Rightarrow \rho + \sigma(\rho''\sigma + \rho'\sigma') = 0$$

$$\Rightarrow \frac{\rho}{\sigma} + (\rho''\sigma + \rho'\sigma') = 0 \quad \text{as } \sigma \neq 0 \qquad (2)$$

$$\Rightarrow (\alpha + \rho N + \rho'\sigma B)' \underset{(1),(2)}{=} 0 \Rightarrow (\alpha + \rho N + \rho' \sigma B) = c \quad \text{for some } c$$

$$\Rightarrow (\alpha - c)(\alpha - c) = (-\rho N - \rho'\sigma B).(-\rho N - \rho' \sigma B)$$

$$= \rho^2 + (\rho'\sigma)^2$$

$$= r^2$$

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Show that the curvature of a regular curve in $\mathbb{R}^3$ is given by

$$\kappa^2 v^4 = \|\alpha''\|^2 - (dv/dt)^2$$

We know that $\kappa = \|\alpha' \times \alpha''\| / \|\alpha'\|^3 \Rightarrow v^4 \kappa^2 = \dfrac{\|\alpha' \times \alpha''\|^2}{v^2}$

Recall that $\|v \times w\|^2 = (v \cdot v)(w \cdot w) - (v \cdot w)^2$ for any vector

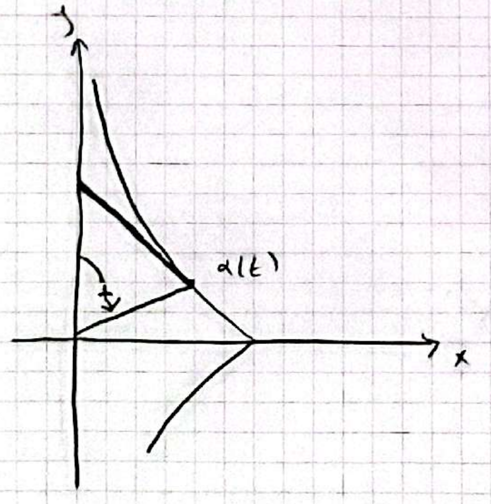
$$\Rightarrow \kappa^2 v^4 = \frac{1}{v^2}\left((\alpha' \cdot \alpha')(\alpha'' \cdot \alpha'') - (\alpha' \cdot \alpha'')^2\right) = \alpha'' \cdot \alpha'' - \frac{(\alpha' \cdot \alpha'')^2}{v^2}$$

$$= \|\alpha''\|^2 - \left(\frac{dv}{dt}\right)^2$$

$$\frac{dv}{dt} = \frac{d(\alpha' \cdot \alpha')^{1/2}}{dt} = \frac{1}{2v} 2\alpha'' \cdot \alpha' = \frac{\alpha' \cdot \alpha''}{v}$$

(by the chain rule)

Let $\alpha(t) : (0, \pi) \to \mathbb{R}^2$ given by $\alpha(t) = (\sin t, \cos t + \log \tan \frac{t}{2})$, where t is the angle that the y-axis makes with the vector $\alpha(t)$.



Show that

(a) α is differentiable regular curve except at $t = \pi/2$

Clearly $\sin t$ and $\cos t + \log \tan \frac{t}{2}$ are diff. on $(0, \pi)$.

$$\alpha'(t) = \left(\cos t, -\sin t + \frac{\frac{1}{2}\frac{1}{\cos^2 t/2}}{\tan \frac{t}{2}}\right)$$

$$= \left(\cos t, -\sin t + \frac{1}{\underbrace{2\sin\frac{t}{2}\cos\frac{t}{2}}_{\sin t}}\right)$$

$$= \left(\cos t, \frac{\cos^2 t}{\sin t}\right)$$

$\alpha'(t) \neq 0$ whenever $t \neq \pi/2$ and $\alpha'(\pi/2) = 0$

Therefore, it is regular except $t = \pi/2$

(b) The length of the segment of the tangent of the tractrix between the point of tangency and the y-axis is constantly equal to 1.

For any $t_0 \in (0,\pi)$ s.t. $t_0 \neq \pi/2$, the tangent line at $t=t_0$ is given by

$$y - \cos t_0 - \log \tan \frac{t_0}{2} = \frac{\cos^2 t_0}{\cancel{\cos t_0} \sin t_0} (x - \sin t_0)$$

$$x=0 \Rightarrow y = \log \tan \frac{t_0}{2}.$$

Hence, the line intersects with y-axis at $(0, \log \tan \frac{t_0}{2})$ s.t. the desired length is

$$\left\| (\sin t_0, \cos t_0 + \log) - (0, \log) \right\| = \left\| (\sin t_0, \cos t_0) \right\| = 1.$$

Consider $v=(1,-1,2)$ $p=(1,3,-1)$, $W = xU_1 + x^2U_2 - z^2U_3$.

Compute $\nabla_v W$

Recall: $\nabla_v W = W(p+tv)'(0)$

$p+tv = (1+t, 3-t, -1+2t)$, $W(p+tv) = (1+t)U_1 + (1+t)^2U_2 - (-1+2t)^2U_3$

$W(p+tv)'(0) = U_1(p) + 2U_2(p) + 4U_3(p)$

$V = -yU_1 + xU_3$, $W = \cos x U_1 + \sin x U_2$. Express the followings in terms of covariant derivatives of U_i's

$\nabla_V(z^2W)$

$$\nabla_V(W) = V(p)[\cos x]U_1 + V(p)[\sin x]U_2$$

$$= (-y(-\sin x) + x.0)U_1 + (-y\cos x + x.0)U_2$$

$$= +y\sin x U_1 - y\cos x U_2$$

Recall $\nabla_V(fW) = V[f]W + f\nabla_V W$ for a function f.

$$\Rightarrow \nabla_V(z^2W) = \underbrace{V(p)(z^2)}_{(x2z)}(\cos x U_1 + \sin x U_2) + z^2.(y\sin x U_1 - y\cos x U_2)$$

$$= U_1(2xz\cos x + z^2y\sin x) + U_2(2xz\sin x - z^2y\cos x)$$

$$\nabla_V(\nabla_V W) = \nabla_V(y\sin x U_1 - y\cos x U_2)$$

$$= V[y\sin x]U_1 - V[y\cos x]U_2$$

$$= (-y^2\cos x)U_1 - (y^2\sin x)U_2$$