# Math 371 Recitation 1

1-) Let $V_1 = U_1 - xU_3$, $V_2 = U_2$, $V_3 = xU_1 + U_3$

(a) $V_1(p), V_2(p), V_3(p)$ are linearly independent at each point of $\mathbb{R}^3$

Say $p = (x, y, z)$ is any point in $\mathbb{R}^3$. Then,

$$V_1(p) = (1, 0, -x)_{(x,y,z)}, \quad V_2(p) = (0,1,0)_{(x,y,z)}, \quad V_3(p) = (x, 0, 1)_{(x,y,z)}$$

Let $c_1, c_2, c_3 \in \mathbb{R}$.

$$c_1 V_1(p) + c_2 V_1(p) + c_3 V_1(p) = (c_1 + c_3 x, c_2, -c_1 x + c_3)_{(x,y,z)} = 0_{(x,y,z)}$$

$$\Rightarrow c_2 = 0, \quad c_3 = c_1 x, \quad 0 = c_1 + c_3 x = c_1 - c_1 x^2$$

$$= c_1 \underbrace{(1+x^2)}_{\neq 0}$$

$$\Rightarrow c_1 = 0, c_3 = 0.$$

(b) Express $xU_1 + yU_2 + zU_3$ as a linear combination of $V_1, V_2, V_3$.

$$xU_1 + yU_2 + zU_3 = (x, y, z)_p, \quad V_1(p) = (1, 0, -x)_p, \quad V_3(p) = (x, 0, 1)_p$$

s.t. $(V_1 + xV_3)(p) = (1+x^2, 0, 0)_p = (1+x^2) U_1(p)$

$$\Rightarrow U_1 = \frac{V_1}{1+x^2} + \frac{x}{1+x^2} V_3$$

Similarly, $(1+x^2) U_3(p) = (0, 0, 1+x^2)_p = (-xV_1 + V_3)(p)$

Hence, $xU_1 + yU_2 + zU_3 = \frac{xV_1}{1+x^2} + \frac{x^2 V_3}{1+x^2} + yV_2 + z\frac{(-x)V_1}{1+x^2} + zV_3$

$$= \frac{x - zx}{1+x^2} V_1 + yV_2 + \frac{x^2 + z}{1+x^2} V_3.$$

2-) $v=(2,-1,3)$, $p=(2,0,-1)$, compute $v_p[f]$, where

(a) $f=y^2z$

$p+tv=(2+2t,-t,-1+3t)$, $f(p+tv)=t^2(3t-1)$

$$v_p[f]=\frac{d}{dt}\left(f(p+tv)\right)\Big|_{t=0}=\frac{d}{dt}\left(t^2(3t-1)\right)\Big|_{t=0}$$

$$=\left(2t(3t-1)+t^2.3\right)\Big|_{t=0}$$

$$=0.$$

(c) $f=e^x\cos y$

If $v_p=(v_1,v_2,v_3)$, then $v_p[f]=\sum v_i\frac{\partial f}{\partial x_i}(p)$

$$\frac{\partial f}{\partial x}=e^x\cos y,\quad \frac{\partial f}{\partial y}=-e^x\sin y,\quad \frac{\partial f}{\partial z}=0$$

$$\rightarrow v_p[f]=2\,\frac{\partial f}{\partial x}(2,0,-1)+(-1)\frac{\partial f}{\partial y}(2,0,-1)+3\,0$$

$$=2e^2+0+0$$

$$=2e^2$$

3-) Let $V=y^2U_1-xU_3$ and $f=xy$, $g=z^3$. Compute the functions

(a) $V[f]$ It is the function that assigns $V(p)[f]$ to each $p\in\mathbb{R}^3$.

$$\frac{\partial f}{\partial x}=y,\quad \frac{\partial f}{\partial y}=x,\quad \frac{\partial f}{\partial z}=0 \qquad \left(U_i[f]=\partial f/\partial x_i\right)$$

$$\underset{\text{linear wrt both } V \text{ and } f}{V[f]}=(y^2U_1-xU_3)[f]=y^2\underbrace{U_1[f]}_{\frac{\partial f}{\partial x}}-x\underbrace{U_3[f]}_{\frac{\partial f}{\partial z}}=y^2y-x\cdot 0=y^3$$

(c) $V[f^2+g^2] = V[f\cdot f + g\cdot g] = V[f\cdot f] + V[g\cdot g]$

$= V[f]f + f\,V[f] + V[g]g + g\,V[g]$

$= 2\left(V[f]f + V[g]g\right)$

$= 2y^3 xy + z^3(-3xz^2)$

$= 2y^4x - 3xz^5$

$V[g] = y^2 U_1[g] - xU_3[g]$

$= y^2\cdot 0 - x\, 3z^2$

$= -3xz^2$

4.) Evaluate the followings

(a) $(z^2-1)dx - dy + x^2 dz$ on $v_p$, where $v=(1,2,-3)$ and $p=(0,-2,1)$.

Recall $dx_i(v_p)=v_i$ where $v=(v_1,v_2,v_3)$ and

$$(f\phi)(v_p) := f(p)\phi(v_p), \quad (\phi_1+\phi_2)(v) = \phi_1(v)+\phi_2(v)$$

$$(z^2-1)dx(v_p) = (p_3^2-1)\cdot v_1 = (1^2-1)\cdot 1 = 0$$

$$dy(v_p) = v_2 = 2, \quad x^2 dz(v_p) = p_1^2 \cdot v = 0$$

$$((z^2-1)dx - dy + x^2 dz)(v_p) = ((z^2-1)dx)(v_p) - dy(v_p) + (x^2 dz)(v_p) = -2$$

$$= (p_3^2-1)v_1 - v_2 + p_1^2 v_3$$

(b) $\phi = x^2 dx - y^2 dz$ on $W = xy(U_1-U_3) + yz(U_1-U_2)$

Recall: $\phi(fV+gW) = f\phi(V)+g\phi(W)$,

$(f\phi+g\psi)(V) = f\phi(V) + g\psi(V)$

At $p$, $\phi(V)$ is $\phi(V(p))$

$dx(U_1)(p) = dx(1,0,0)_p = 1$

We know that $dx(U_1)=1$, $dx(U_2)=0$, $dx(U_3)=0$, $dz(U_1)=0$, $dz(U_2)=0$, $dz(U_3)=1$.

$$\phi(W) = x^2\big(xy(dx(U_1)-dx(U_3)) + yz(dx(U_1)-dx(U_2))\big)$$
$$-y^2\big(xy(dz(U_1)-dz(U_3)) + yz(dz(U_1)-dz(U_2))\big)$$

$$= x^2(xy+yz) - y^2(-xy) = x^3y + x^2yz + xy^3$$

5) A point at which $df$ is zero is called a critical point of $f$. Prove that $p$ is a critical point of $f$ iff

$$\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0.$$

Find all critical points of $f=(1-x^2)y+(1-y^2)z$.

If $v=(v_1,v_2,v_3)$, then

$$df(v_p)=\frac{\partial f}{\partial x}(p)\,dx(v_p)+\frac{\partial f}{\partial y}(p)\,dy(v_p)+\frac{\partial f}{\partial z}(p)\,dz(v_p)$$

$$=\frac{\partial f}{\partial x}(p)\,v_1+\frac{\partial f}{\partial y}(p)\,v_2+\frac{\partial f}{\partial z}(p)\,v_3 \qquad (\star)$$

If $\frac{\partial f}{\partial x}(p)=\frac{\partial f}{\partial y}(p)=\frac{\partial f}{\partial z}(p)=0$, then $df(v_p)=0$ for any tangent vector $v_p$ by $(\star)$. Hence, $p$ is a critical point of $f$.

Conversely, if $p$ is a critical point of $f$, then $df(v_p)=0$ for any tangent vector $v_p$. In particular, for $v=(1,0,0)$, $\frac{\partial f}{\partial x}(p)=0$, for $v=(0,1,0)$, $\frac{\partial f}{\partial y}(p)=0$ and for $v=(0,0,1)$, $\frac{\partial f}{\partial z}(p)=0$.

6) Critical points of $f=(1-x^2)y+(1-y^2)z$

$$\frac{\partial f}{\partial x}=-2xy, \quad \frac{\partial f}{\partial y}=(1-x^2)-2yz, \quad \frac{\partial f}{\partial z}=1-y^2$$

We must solve $\quad -2xy=1-x^2-2yz=1-y^2=0$

$1-y^2=0 \Rightarrow y=+1$ or $y=-1$. $\quad -2xy=0 \Rightarrow x=0$.

$1-x^2-2yz=1-2yz=0 \Rightarrow z=\frac{1}{2y}$ Critical points are $(0,+1,\frac{1}{2})$ and $(0,-1,-\frac{1}{2})$.

7) Let $\phi = dx/y$, $\psi = z\,dy$. Check the Leibnitz formula $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$ in this case by computing each term seperately. Recall $d(\sum f_i dx_i) = \sum (df_i \wedge dx_i)$

$$\phi \wedge \psi = \frac{z}{y}\, dx\, dy \;,\quad d(\phi \wedge \psi) = \frac{1}{y}\, dz\, dx\, dy = \frac{1}{y}\, dx\, dy\, dz$$

$$d\phi = \frac{-1}{y^2}\, dy\, dx = \frac{1}{y^2}\, dx\, dy \;,\quad d\phi \wedge \psi = \frac{1}{y}\, dx\, dy\, dy = 0$$

$$d\psi = d(z\, dy) = dz\, dy = -dy\, dz \,,\quad \phi \wedge d\phi = -\frac{1}{y}\, dx\, dy\, dz$$

$$= d(\phi \wedge \psi).$$

8-) For any function $f$ show that $d(df) = 0$. Deduce that $d(f\,dg) = df \wedge dg$.

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$d(df) = d(f_1) \wedge dx + d(f_2) \wedge dy + d(f_3) \wedge dz$$

$$= (f_{11} dx + f_{12} dy + f_{13} dz) \wedge dx + (f_{21} dx + f_{22} dy + f_{23} dz) \wedge dy + (f_{31} dx + f_{32} dy + f_{33} dz) \wedge dz$$

$$= \underbrace{f_{12}\, dy\, dx}_{-f_{12} dx dy} + \underbrace{f_{13}\, dz\, dx}_{-f_{13} dx dz} + f_{21}\, dx\, dy + \underbrace{f_{23}\, dz\, dy}_{-f_{23} dy dz} + f_{31}\, dx\, dz + f_{32}\, dy\, dz$$

$$= (f_{21} - f_{12})\, dx\, dy + (f_{31} - f_{13})\, dx\, dz + (f_{32} - f_{23})\, dy\, dx$$

Since $f$ is a differentiable function, $f_{21} - f_{12} = f_{31} - f_{13} = f_{32} - f_{23} = 0$

So that $d(df) = 0$. Lastly, $d(f\,dg) = df \wedge dg + f\, d(dg)$
$= df \wedge dg$