Chapter 1

November 5, 2018

1 Field

Definition *A field is a set F, together with two mappings of $F \times F \to F$, called addition and multiplication, written as $(a, b) \to a+b$ and $(a, b) \to ab$ respectively with the following properties:*

(A1) $a + b = b + a$ for all $a, b \in F$ (commutativity)

(A2) $a + (b + c) = (a + b) + c$ for all $a, b, c \in F$ (associativity)

(A3) There is an element in F, denoted by 0_F, such that $a + 0_F = a$ $\forall a \in F$ (additive identity)

(A4) For each $a \in F$ there is an element in F, denoted by $-a$, such that $a + (-a) = 0_F$ (additive inverse)

(M1) $ab = ba$ for all $a, b \in F$ (commutativity)

(M2) $a(bc) = (ab)c$ for all $a, b, c \in F$ (associativity)

(M3) There is an element in F, denoted by 1_F, such that $a1_F = a\ \forall a \in F$ (multiplicative identity)

(M4) For each $a \neq 0_F$ there is an element in F, denoted by a^{-1}, such that $aa^{-1} = 1_F$ (multiplicative inverse)

(D1) $a(b+c) = ab + ac \quad \forall a, b, c$ (distributive law).

Example: Set of real numbers $\mathbb{R}$ with standard addition and multiplication.

Example: Set of binary numbers with modulo 2 addition and multiplication.

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Example: Let $F = \mathbb{R} \times \mathbb{R}$. Let us define $\cdot$ and $+$ as:

$x + y := (x_1 + y_1, x_2 + y_2)$,

$x \cdot y := (x_1y_1 - x_2y_2, x_1y_2 + x_2y_1)$,

where $x = (x_1, x_2) \in F, y = (y_1, y_2) \in F$.

Exercise: Let $F = (0, \infty) = \mathbb{R}_+$ (positive real numbers) Given $x + y := xy$, $x \cdot y := e^{\ln(x)\ln(y)}$, show that F satisfies the axioms of field. Find 1_F and 0_F.

Question: Are polynomials a field? Are matrices a field?

2 Linear Spaces

Definition *A linear space (vector space) V is a set, whose elements are called vectors associated with a field F, whose elements are called scalars. Vectors can be added and they can be multiplied by scalars. These operations satisfy the following properties:*

Vector addition: $x + y, \quad + : V \times V \to V$

(A1) $x + y = y + x \quad \forall x, y \in V$ (commutativity)

(A2) $x + (y + z) = (x + y) + z \quad \forall x, y, z \in V$ (associativity)

(A3) $x + 0 = x \quad \forall x \in V$ (additive identity)

(A4) $x + (-x) = 0 \quad \forall x \in V$ (additive inverse)

Scalar multiplication: $ax, \quad \cdot : F \times V \to V$

(M1) $a(bx) = (ab)x$ for all $a, b \in F$, $x \in V$ (associativity)

(M2) $a(x + y) = ax + ay$ for all $a \in F$, $x, y \in V$ (distributive)

(M3) $(a + b)x = ax + bx$ for all $a, b \in F$, $x \in V$ (distributive)

(M4) $1x = x$ (unit rule)

Example: Show that $0x = 0$

Proof:

Example: (linear space) Set of all vectors of the form v=$(a_1, a_2, \ldots, a_n)$ with $a_i \in F$.

Addition, multiplication are defined componentwise. This space is denoted as F^n.

Let $x, y \in F^n$ $\quad x = (a_1, a_2, \ldots, a_n)$, $y = (b_1, b_2, \ldots, b_n)$

Addition: $x + y := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

Multiplication: $cx := (ca_1, ca_2, \ldots, ca_n)$

Most common examples are $\mathbb{R}^n$ and $\mathbb{C}^n$.

Example: Set of all real valued functions $t \to f(t)$ defined on the real line $F = \mathbb{R}$.

Example: Set of all polynomials with degree n with coefficients in F.

Example: Set of all polynomials with degree less than n with coefficients in F.

Definition *Let V be a linear space defined over field F, denoted by (V, F). A subset W of V is called a subspace if sums and scalar multiples of elements of W belong to W. That is,*

(S1) $w_1 + w_2 \in W \quad \forall w_1, w_2 \in W$

(S2) $cw \in W \quad \forall w \in W$ and $\forall c \in F$

Subset has to be closed under addition and scalar multiplication. All other properties are inherited from the original linear space.

Example: linear space $V = \mathbb{R}^2$, subspace $W = [a\ 0]^T : a \in \mathbb{R}$

Example: linear space $V = \mathbb{R}^2$, subspace $W = [a\ 1]^T : a \in \mathbb{R}$

Example: linear space $V =$ set of all real valued functions $t \to f(t)$;

subspace $W_1 =$ set of all continuous functions.

subspace $W_2 =$ set of all functions periodic with π.

Remark: 0 vector itself is a subspace and it is the smallest subspace.

Definition *The sum of two subsets Y and Z of a linear space X, denoted as $Y + Z$, is the set of all vectors of form $y + z$, where $y \in Y$, $z \in Z$.*

Example: Show that $Y + Z$ is a linear subspace of X, if Y and Z are

Proof:

Example: Prove that if Y and Z are subspaces of linear space X, so is their intersection $Y \cap Z$.

Proof:

Example: If Y and Z are subspaces of a linear space X, is their union $Y \cup Z$ a subspace?

Definition *Let (V, F) and (W, F) be two linear spaces defined over the same scalar field F. The product space of (V, F) and (W, F) is defined as*

- $V \times W = \{(v, w) : v \in V, w \in W\}$

- $(v, w) + (x, y) := (v + x, w + y)$ (vector addition)
- $a(v, w) := (av, aw)$ (scalar multiplication) .

Definition *A* <u>*linear combination*</u> *of n vectors $x_1, x_2, , \ldots, x_n$ of a linear space C is a vector of the form* $a_1x_1, +a_2x_2+, \ldots, +a_nx_n,$ *where a_i's are scalars in F.*

The set of <u>all</u> linear combinations of $x_1, x_2, , \ldots, x_n$ is called the span of $\{x_1, x_2, , \ldots, x_n\}$; denoted by $sp\{x_1, x_2, , \ldots, x_n\}$.

Definition *Vectors $x_1, x_2, , \ldots, x_n$ in X are said to be* <u>*linearly independent*</u> *iff* $a_1x_1, +a_2x_2+, \ldots, +a_nx_n, = 0$ *implies $a_i = 0, \forall i$. Otherwise, they are* <u>*linearly dependent*</u>*.*

Example:

Example: Consider the linear space of polynomials with degree $n \leq 2$. Let subset S=$\{P_1, P_2, P_3\}$ be such that $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$, $\quad \forall t$

Is this set linearly independent?

Example: S=$\{\cos(t), \sin(t), \cos(t - \pi/3)\}$

Definition *Let V be a linear space and (finite) set of vectors S=$\{x_1, \ldots, x_n\}$ be a subset of V. S is said to be a basis for V iff*

- $Span(S) = V$
- S is a linearly independent set.

A (finite dimensional) linear space V has many bases. All these bases must have the same number of vectors. That number is called the dimension of V.

Example: $V = \mathbb{R}^2$, consider the two bases:

$$S1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \qquad S2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

Definition *An* ***ordered basis*** *is a basis $(x_1, x_2, \ldots, x_n)$, where basis vectors are given in a specific ordering.*

If $(x_1, x_2, \ldots, x_n)$ is an ordered basis of V and $y \in V$, then there is a unique n-tuple of scalars $(a_1, a_2, \ldots, a_n)$ such that $y = \sum_{i=1}^{n} a_i x_i$.
Scalars $(a_1, a_2, \ldots, a_n)$ are called the components of y with respect to the ordered basis $(x_1, x_2, \ldots, x_n)$.

Definition *Given n-dimensional linear space V over (field) $\mathbb{R}$, let $B = \{b_1, b_2, \ldots, b_n\}$ be an ordered basis for V. Suppose a vector of $v \in V$ satisfies $v = \sum_{i=1}^{n} \alpha_i b_i$, where $\alpha_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$, then the coefficients α_i are called the* ***coordinates*** *of v w.r.t. basis B.*

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \in \mathbb{R}^n \qquad \text{we use notation } \alpha = [v]_B.$$

Example: Let $\mathbb{R}^{2\times 2}$ denote the linear space of real valued 2×2 matrices. Addition and scalar multiplication are defined as:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix},$$

$$\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}.$$

Example: V : linear space of polynomials with degree $d \leq 2$. Let basis B=$\{P_1, P_2, P_3\}$ be such that $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2 \forall t$. Given $p \in V$ with $p(t) = a + bt + ct^2$, we can write $[p]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

Example: With respect to some ordered basis $B_1 = (x_1, x_2)$ of $\mathbb{R}^2$, let the vectors y_1, y_2, y_3 be presented by $[y_1]_{B_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $[y_2]_{B_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $[y_3]_{B_1} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

That is, $y_1 = 1x_1 + 1x_2$, $y_2 = 1x_1 + 0x_2$, $y_3 = 2x_1 + 3x_2$.

Let our new basis be $B_2 = (y_1, y_2)$. Express y_3 w.r.t. this new basis.

Remark: For a given ordered basis, the representation of a vector is unique.

Exercise: Prove the remark.

Theorem *Let V be an n-dimensional linear space over $\mathbb{R}$. Let B_1 and B_2 be two bases for V, then there exist $n \times n$ real matrix P such that $[x]_{B_1} = P[x]_{B_2} \quad \forall x \in V$.*

Proof:

Claim: Matrix P is invertible.

3 Normed linear spaces

Consider a linear space V over F, where F is either $\mathbb{R}$ or $\mathbb{C}$. Let there be a function $x \to \|x\|$ that assigns to each $x \in V$, a nonnegative real number $\|x\| \in \mathbb{R} \geq 0$. Such function is called a **norm** if it satisfies the following properties.

(P1) $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$

(P2) $\|\alpha x\| = |\alpha| \, \|x\| \; \forall x \in V$ and $\alpha \in F$

(P3) $\|x\| = 0 \Leftrightarrow x = 0$

The expression "‘$\|x\|$"’ is read "‘the norm of x"’ and the function $\|.\|$ is said to be a norm on V. The triplex $(V, F, \|.\|)$ is called a **normed space**.

Remark: Perhaps the most important use of the norm is that through it, we can quantify "‘distance"’ between two points in our linear space. Namely, the distance between $x_1, x_2 \in V$ is the norm of the vector $x_1 - x_2$ or $x_2 - x_1$: $\|(x_1 - x_2)\|$. Since $x = x - 0$, the norm of x, $\|x\|$ is the distance of x to the origin 0. With a proper tool for measuring distance (norm), one can begin studying the "‘geometry "’ of the space (parallelism, orthogonality, area, volume, shape in general).

Example: Let $V = \mathbb{R}^2$, $F = \mathbb{R}$, and let $x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$.

i) $\|x\|_1 := |\alpha_1| + |\alpha_2|$. Is $\|.\|_1$ a norm?

ii) $\|x\|_2 := (|\alpha_1|^2 + |\alpha_2|^2)^{\frac{1}{2}}$. Is $\|.\|_2$ a norm?

iii) $\|x\|_\infty := \max(|\alpha_1|, |\alpha_2|)$. Is $\|.\|_\infty$ a norm?

All these norms can be generalized into what we call a **'p-norm'**.

$$\|x\| := (|\alpha_1|^p + |\alpha_2|^p)^{\frac{1}{p}}$$

.

Note that $\lim_{p-\to\infty} \|x\|_p = \|x\|_\infty$.

Geometric visualization of these norms:

Example:

Example:

Special cases:

4 Matrix Norms

Example: Let $V = \mathbb{R}^{n\times m}$, and $A = [a_{ij}]$,

$$\|A\| = \max_{i,j} |a_{ij}| \text{ is a norm.}$$

Example: Let $V = \mathbb{R}^{n\times m}$, and $A = [a_{ij}]$,

$$\|A\| = \max_{i} \sum_{j=1}^{n} |a_{ij}| \text{ (abs sum of rows)}$$

Exercise: Show that this is a norm.

There are different ways to define matrix norms. One way of defining matrix norms is to consider the matrix as a vector.

Example:

Definition *$A : \mathbb{R}^n \to \mathbb{R}^m$ be an $m \times n$ matrix. Let $\|.\|_{\mathbb{R}^n}$ and $\|.\|_{\mathbb{R}^m}$ denote the norms (vector norms) in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. The* ***induced norm*** *of a matrix is defined as*

$$\|A\| := \max_{x \in \mathbb{R}^n, x \neq 0} \frac{\|Ax\|_{\mathbb{R}^m}}{\|x\|_{\mathbb{R}^n}}.$$

Remark: The induced matrix norm is defined in terms of vector norms. An equivalent definition is:

$$\|A\| := \max_{\|x\|=1} \|Ax\|.$$

Remark: $\|Ax\| \leq \|A\| \|x\|$ (for induced norms).

Furthermore, there exists a vector x^* such that $\|Ax^*\| = \|A\| \|x^*\|$ which may not be unique.

Example: Choose $\|.\|_2$ in $\mathbb{R}^n$ and $\mathbb{R}^m$,

$$\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \sqrt{(Ax)^T Ax} = \max_{\|x\|=1} \sqrt{x^T A^T Ax}$$

Example: Choose $\|.\|_\infty$ as the norm in $\mathbb{R}^n$ and $\mathbb{R}^m$ and find the induced norm $\|A\|$

5 Convergence

Definition *Let $(V, F, \|.\|)$ be a normed space. Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of vectors in V. $v_n \in V$ $n = 1, 2, \ldots$. The sequence is said to be* ***convergent*** *to the limit $\bar{v} \in V$ iff $\|v_n - \bar{v}\| \to 0$ as $n \to \infty$*

OR: given any $\epsilon > 0$, $\exists N$ (N depends on ϵ) such that $n \geq N$ implies $\|v_n - \bar{v}\| \leq \epsilon$.

Remark: A sequence that is not convergent is called **divergent**.

Example: Given $V = \mathbb{R}$ and $\|v\| = |v|$, consider the sequence $\left\{\left(\frac{1}{2}\right)^n\right\}_{n=1}^{\infty}$.

Example: Consider the sequence $\{(-1)^n\}_{n=1}^{\infty}$

Note: In most engineering applications, we are interested in the convergence of an iterative algorithm. In general the problem is, we do not know where! The problem with the given definition of the convergence is that it requires the limiting element $\bar{v} \in V$, an element we may not know, to verify convergence. An attempt to get rid of $'\bar{v} - dependence'$ resulted in the following concept.

Definition *Let $(V, F, \|.\|)$ be a normed space. A sequence $\{v_n\}_{n=1}^{\infty}$ in V is said to be a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N$(depending on ϵ) such that $\|v_n - v_m\| < \epsilon$ for all $n, m > N$.*

Remark: Every convergent sequence is a Cauchy sequence. The converse, in general, is not true.

Example: Consider the normed space $(\mathbb{Q}, \mathbb{Q}, |.|)$, (i.e., set of rational numbers over the field rational numbers with norm being the absolute value). Is the sequence $\left\{1 + \sum_{i=1}^{n} \frac{1}{i!}\right\}_{n=1}^{\infty}$ convergent?

Definition *A normed space is said to be **complete** if every Cauchy sequence is convergent. A complete normed space is called a **Banach Space**.*

Example: "'A normed space that is not complete"'

Let $V = \{f | f : [-1,1] \to \mathbb{R},$ f is continuous and $\int_{-1}^{1} |f(t)| dt < \infty\}$. Define $\|f\|_1 := \int_{-1}^{1} |f(t)| dt$.

Now consider the sequence $\{f_n\}_{n=1}^{\infty}$ defined as follows:

6 Inner Product Space

An inner product space is a linear space with an additional structure called inner product.

Definition *Let V be a linear space over field F. An inner product is a map of the form $\langle \cdot, \cdot \rangle : V \times V \to F$ that satisfies the following three axioms for all vectors $x, , y, z \in V$ and all scalars $\alpha \in F$.*

1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)

2) a) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ (linearity in the first argument)

   b) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ (linearity in the first argument)

3) $\langle x, x \rangle \geq 0$ with equality only for $x = 0$ (positive defineteness)

Note that:

$$\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

and, $$\langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$$

Example: $V = \mathbb{C}^n$, $\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$

Example:

Theorem *"'Cauchy-Schwarz inequality"'*

$$| \langle x, y \rangle |^2 \leq \langle x, x \rangle \langle y, y \rangle$$

Proof:

Remark: An inner product induces a norm defined as

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Theorem $\sqrt{\langle x, x \rangle}$ *is a norm.*

Proof:

Remark: Every inner product space is a normed space. Converse is not true.

Definition *An inner product space that is complete with respect to the norm induced by the inner product is called a* ***Hilbert Space****.*

Definition *"'Orthogonality"'*

*Two vectors $x, y \in V$ are said to be **orthogonal** if $\langle x, y \rangle = 0$. Likewise, two subsets $S_1, S_2 \subset V$ are said to be **orthogonal** if $\langle x, y \rangle = 0$ for all $x \in S_1$ and $y \in S_2$.*

Example: $V = \mathbb{R}^n$, $\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$ or ($\langle x, y \rangle = x^T y$ where $x = [x_1, x_2, \ldots, x_n]^T$). Consider the canonical basis set $B = \{e_1, e_2, \ldots, e_n\}$ where $e_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T$. Then we have $\langle e_i, e_j \rangle = 1$ if $i = j$ and $\langle e_i, e_j \rangle = 0$ otherwise, i.e., $\langle e_i, e_j \rangle = \delta_{ij}$.

Example: $V = \{f | f : [-\pi, \pi] \to \mathbb{C}$ such that $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$, and f is continuous $\}$

Let $\langle f_1, f_2 \rangle = \int_{-\pi}^{\pi} f_1(t)\overline{f_2(t)}dt$ be the inner product.

Consider the set of vectors $\{e^{jnt}\}_{n=-\infty}^{\infty}$

Question: Let V be an inner product space, Let $S_1 = \{v_1, v_2, \ldots, v_n\}$ be a linearly independent set. Can we obtain another set $S_2 = \{w_1, w_2, \ldots, w_n\}$ such that

i) $\text{Span}(S_2) = \text{Span}(S_1)$

ii) The vectors in S_2 are pairwise orthogonal. That is, $\langle w_i, w_j \rangle = 0 \quad \forall i \neq j$?

7 Gram-Schmidt Orthogonalization

Step 1: $w_1 := v_1$

Step 2: $w_2 := v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$

Step 3: $w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$

$\vdots$

Step n: $w_n := v_n - \sum_{i=1}^{n-1} \frac{\langle v_n, w_i \rangle}{\langle w_i, w_i \rangle} w_i$

Proof:

8 Linear Mapping

Definition *Let V and W be linear spaces over the same field F. A linear transformation $\mathcal{T}$ is a mapping $\mathcal{T} : V \to W$ satisfying*

$$\mathcal{T}(a_1 x_1 + a_2 x_2) = a_1 \mathcal{T}(x_1) + a_2 \mathcal{T}(x_2) \qquad \forall a_1, a_2 \in F \quad \textit{and} \quad \forall x_1, x_2 \in V$$

Example: $V = W$ polynomials of degree less than n in S; $\mathcal{T} = \frac{d}{ds}$

Example: $V = W = \mathbb{R}^2$. Let $\mathcal{A}$ be defined as,

$$\mathcal{A}x = \begin{bmatrix} \alpha_1 \\ \\ \alpha_2 + \alpha_1 \end{bmatrix}, \quad \text{where} \quad x = \begin{bmatrix} \alpha_1 \\ \\ \alpha_2 \end{bmatrix}.$$

Example:

Example: $V = W = \mathbb{R}^2$. If $\mathcal{T}$ is a rotation around the origin by an angle θ then $\mathcal{T}$ is a linear mapping.

Example: $V = W =${continuous functions of type $f : [0,1] \to \mathbb{R}$}; $\mathcal{T}_f = \int_0^1 f(s)ds$

Definition *Given linear transformation $\mathcal{T} : V \to W$, the **null space** of $\mathcal{T}$ is the set of all $x \in V$ satisfying $\mathcal{T}x = 0_w$. That is,*

$\mathcal{N}(\mathcal{T}) := \{x \in V : \mathcal{T}x = 0\}$

Definition *Given linear transformation $\mathcal{T} : V \to W$, the **range space** of $\mathcal{T}$ the set of all $w \in W$ satisfying $\mathcal{T}v = w$. That is,*

$\mathcal{R}(\mathcal{T}) := \{w \in W : w = \mathcal{T}v \quad \text{for some} \quad v \in V\}$

Claim: For a linear transformation $\mathcal{T} : V \to W$, $\mathcal{N}(\mathcal{T})$ is a linear subspace of V.

Proof:

<u>Claim:</u> $\mathcal{R}(\mathcal{T})$ is a subspace of W.

Proof:

Definition *A linear transformation* $\mathcal{T} : X \to Y$ *is* ***one-to-one*** *if* $x_1 \neq x_2 \Rightarrow \mathcal{T}(x_1) \neq \mathcal{T}(x_2)$

Example:

Example:

Theorem *Let* $\mathcal{T} : V \to W$ *be a linear transformation. Then mapping* $\mathcal{T}$ *is one-to-one if and only if* $\mathcal{N}(\mathcal{T}) = \{0\}$.

Proof:

Definition *A linear transformation $\mathcal{T} : X \to Y$ is* ***onto*** *if $\mathcal{R}(\mathcal{T}) = Y$, otherwise it is called* ***into****.*

Example:

Example:

9 Matrix representation of a linear transformation

Let V (dim(V)=n), and W (dim(W)=m) be two linear spaces over the same field F. Let $\mathcal{A} : V \to W$ be a linear transformation. Given two bases $B = (v_1, v_2, \ldots, v_n)$ and $C = (w_1, w_2, \ldots, w_n)$, transformation $\mathcal{A}$ can be represented by an $m \times n$ matrix $[a_{ij}]$ ($a_{ij} \in F$) such that for every $v \in V$ and $w \in W$ satisfying $w = \mathcal{A}v$, we have $[w]_C = [a_{ij}][v]_B$.

$m \times n$ matrix $\mathbf{A}$ is called the matrix representation of $\mathcal{A} : V \to W$ with respect to the basis sets B and C.

Example: Let $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ be a $90°$ counter-clockwise rotation about the origin. Show that $\mathcal{A}$ is linear. Find the matrix representation of $\mathcal{A}$ for

i) $B_1 = (v_1, v_2)$

ii) $B_2 = (v_1, v_3)$ where,

Example: $V = \mathbb{R}^{2\times 2}$ (space of 2×2 real matrices) .

Let linear map $\mathcal{A} : V \to V$ be $\mathcal{A}v = Sv + vS^T$, where $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ for the basis $B = \bar{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Find the matrix representation of $\mathcal{A}$.

Example: $V = \{f : f : \mathbb{R} \to \mathbb{R}, f \text{ is a polynomial with deg} < 3\}$. $\mathcal{A} : V \to V$ is defined as $\mathcal{A}(f) = f'$ (the derivative of f). Let basis the be $B = \{1, s, s^2, s^3\}$. Find the matrix $\mathbf{A}$.

Given the matrix representation of a linear transformation $\mathcal{A}_t : V \to W$ with respect to the basis sets B and C. One can draw the following diagram considering a change of basis.

<u>Special case:</u> If $W = V$, that is $\mathcal{A}_t$ is a map from V to itself, $\mathcal{A}_t : V \to V$, then bases B, C, $\bar{B}$ and $\bar{C}$ can be chosen to be the same. Then we have:

Example: Consider the 90° counter-clockwise rotation example; find $P \in \mathbb{R}^{2\times 2}$ such that $[v]_{B_1} = P[v]_{B_2}$ and verify that $A_2 = P^{-1}A_1P$.

10 Range and Null Space

Let $\mathcal{A} : V \to W$ be a linear map. Let dim(V)$= n$ and Let dim(W)$= m$. Recall that,

- $\mathcal{R}(\mathcal{A})$: range space of $\mathcal{A}$, subspace of $W \Rightarrow$ dim($\mathcal{R}(\mathcal{A})$)$\leq$ m.

- $\mathcal{N}(\mathcal{A})$: null space of $\mathcal{A}$, subspace of $V \Rightarrow$ dim($\mathcal{N}(\mathcal{A})$)$\leq$ n.

Theorem *dim($\mathcal{R}(\mathcal{A})$)+dim($\mathcal{N}(\mathcal{A})$) = dim (V)*

Proof:

Definition *Given linear transformation $\mathcal{A} : V \to W$, let $\boldsymbol{A}$ be the matrix representation (wrt some bases). Then rank of $m \times n$ matrix $\boldsymbol{A}$ is defined as rank($\boldsymbol{A}$)=dim($\mathcal{R}(\mathcal{A})$)*

Fact: Rank($\mathbf{A}$) is equal to maximum number of linearly independent column vectors of $\mathbf{A}$. Rank($\mathbf{A}$) is equal to maximum number of linearly independent row vectors of $\mathbf{A}$.

Example: $\mathcal{A}_t : V \to W$, where dim(V)=2, and dim(W)=4

Example: